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The deformation of granular materials occurs ina variety of applications: soil dynamics; avalanch flows; grain flow in bins. Our goal in this project is to develop a more complete understanding of the mechanics of granular flow. The project consists of two distinct but related phases. One phase is a study of the mathematical structure of constitutive relations used to model the granular medium. The research here is primarily concerned with the stability and well posedness of the evolution equations governing flow. Mostof our efforts have concentrated on the Critical State Theory of Soil Mechanics, a mathematically attractive theory which has some success in modeling soil deformations. In particular, the Critical State Theory is reasonably successful at modelling the deformation of so called "Cam clay", a type of clay tested by the Cambridge soil mechanics group in the late 1950's.

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Final Report: PLASTIC DEFORMATION OF GRANULAR MATERIALS

E. BRUCE PITMAN*

April 1, 1988 - March 31, 1989

INTRODUCTION

The deformation of granular materials occurs in a variety of applications: soil dynamics; avalanch flows: grain flow in bins. Our goal in this project is to develop a more complete understanding of the mechanics of granular flow. The project consists of two distinct but related phases. One phase is a study of the mathematical structure of constitutive relations used to model the granular medium. The research here is primarily concerned with the stability and well posedness of the evolution equations governing flow. Most of our efforts have concentrated on the Critical State Theory of Soil Mechanics, a mathematically attractive theory which has some success in modeling soil deformations. In particular, the Critical State Theory is reasonably successful at modelling the deformation of so called "Cam clay", a type of clay tested by the Cambridge soil mechanics group in the late 1950's. Critical State Theory applied to "Cam clay" is, therefore, a useful model for soil dynamics. Complementing this theoretical study is a computational phase, consisting of numerical simulation of cannonical granular flows. In Section 1 we briefly review the equations governing granular flow; this section includes very preliminary ideas regarding a fully non-linear constitutive relation. In Section 2 we describe the results on the stability of flow. In Section 3 we describe work on the computational phase of this project. A review of many of these issues appears in [8].

§1. EQUATIONS OF MOTION

(a) Critical State Theory. The fundamental equations governing the deformation of a continuum are the balance laws of mass and momentum :

$$d_t\rho + \rho\nabla \cdot v = 0 \tag{1.1}$$

$$\rho d_t v + \nabla \cdot T = b \tag{1.2}$$

Here ρ is the density of the material at the point $x = (x_1, x_2, x_3)$ and time t, $v = (v_1, v_2, v_3)$ is the velocity, and T is the symmetric stress tensor. Body forces are represented by b, and $d_t = \partial_t + v \cdot \nabla$ is the convective derivitive. We employ the summation convention throughout this report.

Equations 1.1 - 1.2 represent four equations for the ten unknowns ρ, v_i, T_{ij}. In order to close the system, we need to postulate constitutive laws relating the density and velocities

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to the stresses. In [**9,11**], Pitman and Schaeffer employed the theory of Critical State Soil Mechanics [**3,12**] to provide these relations. If we regard compressive stresses as positive and denote the strain-rate tensor as $V_{ij} = -\frac{1}{2}(\partial_j v_i + \partial_i v_j)$, these constitutive laws may be written:

$$\phi(T, \rho) = 0 \tag{1.3}$$

$$\frac{\partial \phi}{\partial T_{ij}} = \mu V_{ij}. \tag{1.4}$$

Equation 1.3 is the *Plastic Yield Condition* and Equation 1.4 is the *Associated Flow Rule.* In particular, Equation 1.4 states that the normal to the Yield Surface $\phi = 0$ is proportional to the strain-rate tensor. Note, however, that the constant of proportionality, μ, is not specified a *priori* (as in elasticity) but must be determined as part of the solution procedure. Different versions of the theory may be examined by altering the function ϕ. It is usually required that ϕ be a convex function of T and monotone in ρ. This monotonicity generates a nested sequence of yield surfaces in stress space; such a picture lies at the heart of the Critical State Theory. (Remark: For some materials like dry sand, ϕ may be taken to have the form

$$\phi(T, \rho) = \varphi(tr(T), |dev(T)|) - \rho^{\frac{2}{\beta}}.$$

Here, $tr(T)$ denotes the trace of T; $|dev(T)|$ is the Euclidean norm of the deviator of T, where

$$dev(T) = T - \frac{tr(T)}{3} I;$$

and β is a small parameter which measures compressibility, typically $\beta \approx 10^{-1} - 10^{-2}$.)

The Associated Flow Rule 1.4 is not universally accepted in plasticity research, especially metal plasticity [**2**]; associative flow rules were introduced in plasticity theory by analogy with work in elasticity. See subsection (b) for a different approach. Standard non-Associative theories postulate a "flow potential" ψ and the flow rule $\frac{\partial \psi}{\partial T_{ij}} = \mu V_{ij}$. We consider only the Associative theory here.

We claim that the system of equations 1.1 – 1.4 is closed. There are now 11 unknowns, μ being added to the previous list; there are also 11 equations with which to determine these variables. However, Equations 1.3–1.4 are not evolution equations, and the mixed differential/algebraic character makes the system is difficult to analyze. Considerable simplification results if we solve 1.3–1.4 for T and μ as functions of V and ρ and write

$$T = T(V, \rho). \tag{1.5}$$

We remark that in Equation 1.5, T, when considered as a function of V, is homogeneous of degree zero; this homogeneity is characteristic of a rate independent theory.

Before finishing this section, we define some terminology which shall be used subsequently. Let ϵ_i denote the eigenvalues of the strain-rate tensor V. By *fully three dimensional flow* we mean flows for which $\epsilon_i \neq 0$ for $i = 1, 2, 3$. Two dimensional flow, then, means either flows for which one of the $\epsilon_i = 0$ or flow in only two space dimensions x_1, x_2.

(b) Alternative Constitutive Relations.

The Critical State theory predicts the initial deformation of clays and soils reasonably well. It has difficulties, however, with cyclic loading paths and with sandy materials. In addition, when the Critical State theory predicts ill posedness there is no mechanism in the theory which allows for recovery. (We skirt this difficulty by adding kinetic theory terms; see Section 3). This defect is shared by most standard deformation theory models. Preliminary work has begun to investigate a class of "Yield-Vertex" theories of the type introduced by Christoffersen and Hutchinson [2] for polycrystalline metals.

The fundamental relation in the Yield-Vertex models is a fully non-linear stress strain-rate relation of the form:

$$V_{ij} = \Psi_{ij}(T, \dot{T}). \tag{1.6}$$

Here, $\dot{T}$ denotes the Jauman, or co-rotating, time derivitive of the stress T. For a rate-independent theory, Ψ must be homogeneous of degree 1 in $\dot{T}$; symmetry conditions place other constraints on the form of (1.6). Modelling is required in order to correctly account for proportional and non-proportional loading and for unloading in granular materials.

To summarize the work to date, Schaeffer and Shearer have begun an analysis of the structure of the Yield-Vertex theory. In state space, the non-hyperbolic region appears to be of finite extent; it is possible for the linearized equations to become ill posed, but for the system to remain non-linearly stable. Pitman has performed numerical tests on model problems in order to gain an understanding of the dynamics predicted by the theory. A full investigation of the theory is planned.

§2. INSTABILITY

We may substitute Equation 1.5 into 1.1–1.2 to obtain a system of four time dependent partial differential equations in the variables ρ, v_1, v_2, v_3 given by:

$$d_t\rho + \rho\partial_j v_j = 0 \tag{2.1}$$

$$\rho d_t v_i + \left(\frac{\partial T_{ij}}{\partial \rho}\right)\partial_j\rho - \left(\frac{\partial T_{ij}}{\partial V_{kl}}\right)\partial_j\partial_k v_l = 0. \tag{2.2}$$

Here we have dropped body forces from 2.2. Equations 2.1–2.2 formally resemble the Navier-Stokes equations for a viscous compressible fluid. However, this apparent similarity is illusary. The primary difference between the fluid equations and the system above is in the nature of the viscous dissipation term $\frac{\partial T}{\partial V}$. In the Navier-Stokes system, the symbol matrix associated to this dissipation is negative-definite; in Equations 2.1–2.2, the symbol is only semi-definite. The non-definiteness of this symbol is the origin of many of the complex structures in granular flows.

Schaeffer and Pitman [9,11] examined the linear well posedness of Equations 2.1-2.2. In particular, they found that fully three dimensional flows are well posed. However two dimensional flows may be ill posed with a growth rate of $O(|\xi|)$, where ξ is the Fourier variable dual to x. Our studies in this phase of the project have examined the stability of the Critical State equations [10,13]. In this section we summarize those results.

For many typical granular flows, there exist two well separated time scales [**9**]. It is possible to exploit these separate scales and incorporate into our analysis the effects of terms not included in [**9,11**]. To this end, define a new time $\tilde{t} = \epsilon t$ and a similarly scaled velocity $\tilde{v} = \epsilon^{-1} v$. Substitute into Equations 2.1–2.2 and recall the homogeneity of $T(V, \rho)$ to find (dropping the tilde)

$$\partial_t \rho + \partial_j(\rho v_j) = 0 \tag{2.3}$$

$$\epsilon^2 \rho d_t v_i + (\frac{\partial T_{ij}}{\partial \rho})\partial_j \rho - (\frac{\partial T_{ij}}{\partial V_{kl}})\partial_j \partial_k v_l = 0. \tag{2.4}$$

Now make the *quasi-dynamic approximation* by setting $\epsilon = 0$. Linearize the equations about a steady, homogeneous solution ρ_0, v_0. In the linearization, four terms arise from the derivitive $\partial_j(\rho v_j)$; by appropriate change of coordinate systems we can drop all of these terms except $\rho_0 \partial_j v_j$. See [**13**] for details. We derive, then, a system of one evolution equation for ρ coupled to three apparently elliptic equations for the v_i:

$$\partial_t \rho + \rho_0 \partial_j v_j = 0 \tag{2.3*}$$

$$\frac{\partial T_{ij}}{\partial \rho}(\rho_0, V_0)\partial_j \rho - \frac{\partial T_{ij}}{\partial V_{kl}}(\rho_0, V_0)\partial_j \partial_k v_l = 0. \tag{2.4*}$$

Consider exponential solutions of the form

$$\begin{bmatrix} \rho \\ v \end{bmatrix} = exp(i(x, \xi) + \lambda t) \begin{bmatrix} \bar{\rho} \\ \bar{v} \end{bmatrix} \tag{2.5}$$

where $(x, \xi) = x_j \xi_j$. Substituting 2.5 into 2.3*–2.4*, we derive a generalized eigenvalue problem for λ:

$$S(\xi) \begin{bmatrix} \bar{\rho} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \lambda \bar{\rho} \\ 0 \end{bmatrix}. \tag{2.6}$$

Here the principal part of $S(\xi)$ has the block structure

$$\begin{pmatrix} 0 & \xi^T \\ -C(\xi) & Q(\xi) \end{pmatrix} \tag{2.7}$$

where ξ^T denotes the transpose of the column vector ξ. Performing a similarity transformation to eliminate complex entries, $C(\xi)$ takes the form

$$C_i(\xi) = \rho_0 (\frac{\partial T_{ij}}{\partial \rho})\xi_j \tag{2.8}$$

and Q is given by

$$Q_{ik}(\xi) = -(\frac{\partial T_{ij}}{\partial V_{kl}})\xi_j\xi_l \tag{2.9}$$

We call the Equations 2.1–2.2 (linearly) stable if the real part of λ is negative: $Re\,\lambda(\xi) < 0$.

Our first results are in two space dimensions and relate the growth of the generalized eigenvalue λ to the ellipticity of the steady state equations associated with 2.3*–2.4*

$$\rho_0\partial_j v_j = 0 \tag{2.10}$$

$$(\frac{\partial T_{ij}}{\partial \rho})\partial_j\rho - (\frac{\partial T_{ij}}{\partial V_{kl}})\partial_j\partial_k v_l = 0 \tag{2.11}$$

Call 2.10–2.11 *elliptic* if $det S(\xi) \neq 0$ for all nonzero $\xi \in \mathbf{R}^2$. Roughly speaking, the theorem below states that Equations 2.3–2.4 become unstable when 2.10–2.11 change type from elliptic to hyperbolic.

THEOREM 2.1. *In two dimensions, Equations 2.3-2.4 are stable if and only if* (i) *Equations 2.10-2.11 are elliptic; and* (ii) *both eigenvalues of the stress tensor* T *are non-negative.*

(Remark: Shearer and Schaeffer [**14**] have shown that the results of this theorem remain true for the full system 2.1–2.2.)

One implication of this theorem is that Equations 2.3–2.4 may be unstable even when the stresses are on the consolidation side of the yield locus (i.e. stress states for which $tr(V) > 0$) [**9**]. This statement contradicts a widely held view that deformation is stable on the consolidation side of the yield locus and unstable on the expansion side (i.e. $tr(V) < 0$); the results of [**9**] show that the Critical State (i.e. $tr(V) = 0$) is the boundary between well posed and ill posed behavior. A second consequence of the theorem is that all Fourier modes along the specific unstable direction lose stability simultaneously. Which specific mode eventually dominates growth (in the linear regime) depends on initial and boundary conditions; this suggests extreme sensitivity of experiments to imperfections in laboratory protocol.

The analogue of Theorem 2.1 holds in three dimensions, but the geometric interpretation is more complicated [**10**]. In particular, when the steady state equations lose ellipticity they do not become hyperbolic; rather they are of mixed type. Because of the complicated geometry of three dimensional flow, MACSYMA is being employed to perform the algebra of the problem.

§3. NUMERICAL SIMULATIONS

Preliminary efforts at numerically integrating Equations 2.1–2.2 were reported in the original proposal. Here we describe our current work on this phase of the project.

Consider the following scenario suggested by Theorem 2.1 for the development of shear bands. Beginning with a homogeneous stable flow, material consolidates upon deformation, eventually reaching the instability locus. Subsequent deformation remains smooth, but it

is no longer homogeneous and local imperfections grow in time. At some point in the sample, an imperfection ultimately becomes ill posed as it reaches the Critical State locus. Such a point could act as the seed of a velocity discontinuity which then expands into a shear band. Numerical simulations should be able to confirm this conjectured route to shear band formation, but the ill posedness of two dimensional deformations precludes the simple intergration of Equations 2.1–2.2. We follow a suggestion of Jackson and regularize the system.

Let us elaborate. Near a shear band, there is large momentum transfer over distances on the order of a few particle diameters. Jenkins [4], Savage [5], and colleagues have developed a theory for granular flow at very high shear rates based on an extension of standard kinetic theory to slightly inelastic particles. Incorporating the Enskog dense gas correction, the granular continuum equations are essentially the Navier-Stokes equations for a compressible, heat conducting fluid. The "heat" is not molecular energy, but the so-called "granular temperature" measuring a particle's deviation from the mean motion of its neighbors. There is one non-classical term in the theory, a heat sink which represents the energy loss due to inelastic collisions. Let the stress tensor derived from this kinetic theory be T^k, and denote the stress tensor of the Critical State theory as T^f. Jackson [6] considers a system consisting of Equations 1.1–1.2 and the energy equation of the kinetic theory, with the total stress being the sum

$$T = T^f + T^k; \tag{3.1}$$

[6] also shows how to derive appropriate boundary conditions for flow.

The primary consequence of the addition of these kinetic stresses is to add an overall regularization to the system 2.1–2.2. In unpublished notes, Schaeffer and Pitman demonstrate that the system 1.1–1.2, 3.1 is well posed. However, near unstable and ill posed states the system possesses only a small amount of damping of very high frequency Fourier components. For moderate wave numbers near these states there is a competition between the growth of the friction terms and the damping of the kinetic terms.

In order to numerically solve the equations without extreme time-step limitations and to adequately account for the potential growth of Fourier modes as outlined above, we use an implicit Beam-Warming type scheme [1] to integrate the equations. Beam-Warming allows for extension to fully multi-dimensional computations. The hyperbolic terms are treated seperately from the parabolic-like terms. Following Harten and Yee [15] the hyperbolic terms are differenced in such a way as to (try to) preserve the Total Variation Diminishing character of such terms. Parabolic-like terms are centered differenced, and the entire scheme is constructed in "delta" form. The resulting set of difference equations involve block-tridiagonal matrix inversion. Such schemes have been successfully applied to aerodynamic calculations.

Our first numerical experiments were confined to quasi-one-dimensional flow in a hopper. See the attached figure, where we [illegible] the velocity, as a function of position, at various times. The geometry of this problem is similar to that for a spherically expanding wave passing through the ground, and the problem formulation is simpler in the hopper problem. In the simulation, we consider initially steady flow; at time $t = 0$, we suddenly increased the flow rate at the exit. We have examined the transient behavior as flow settles into a

new quiescent steady-state regime. Velocity waves propagate quickly through the material and equilibrate rapidly; the solid line in the figure is the steady state, attained after about 10 seconds. Note the overshoot of the velocity profiles. Density variations remain active for an extended period of time and Stresses become negative during flow, signaling tensile loads. In many granular flow simulations, tensile stresses lead to catastrophic numerical instabilities: the visco-plastic formulation (3.1) allows us to continue computations through the tensile region.

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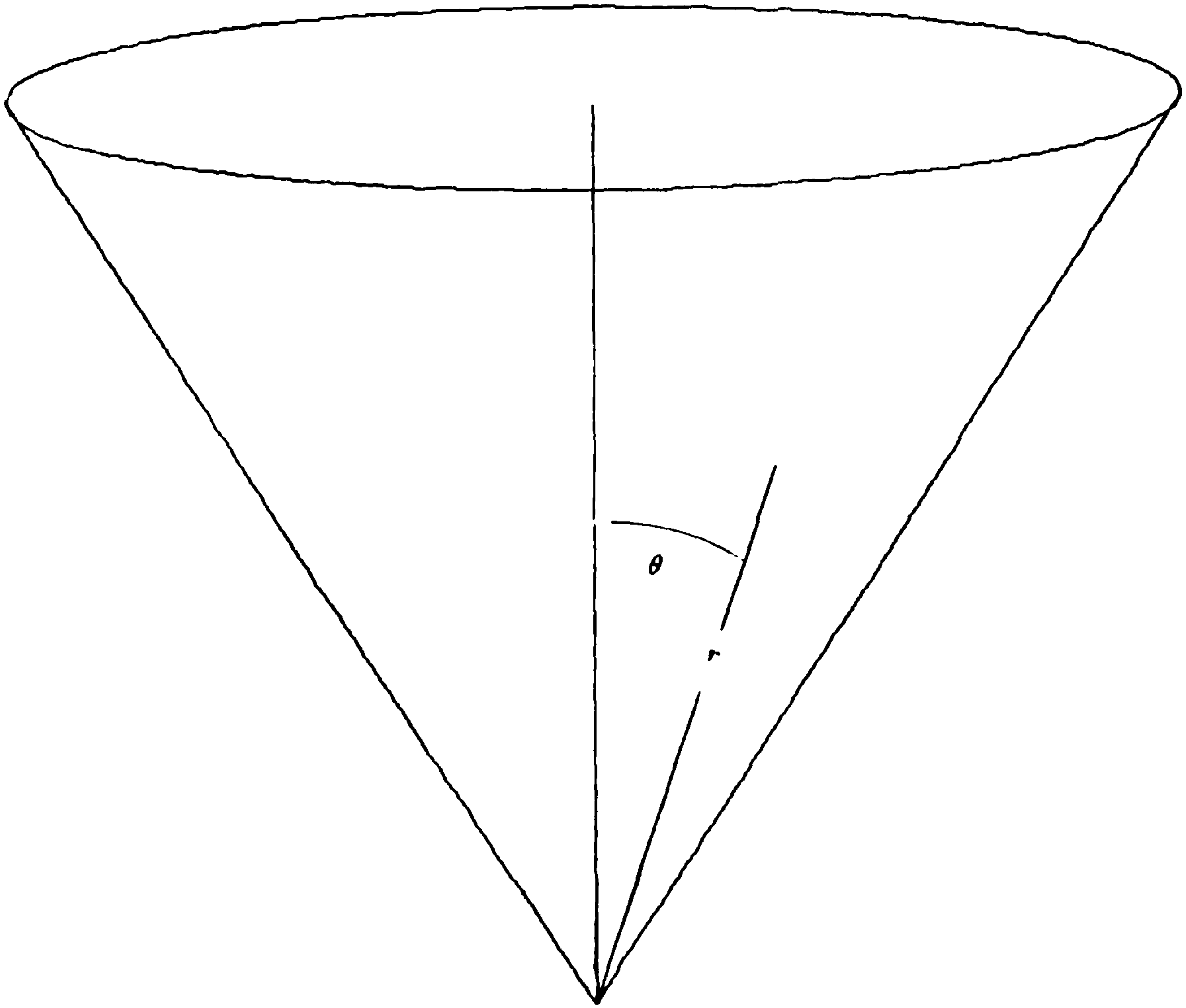
†The research reported in these papers has been conducted as part of this project

Figure 2.1



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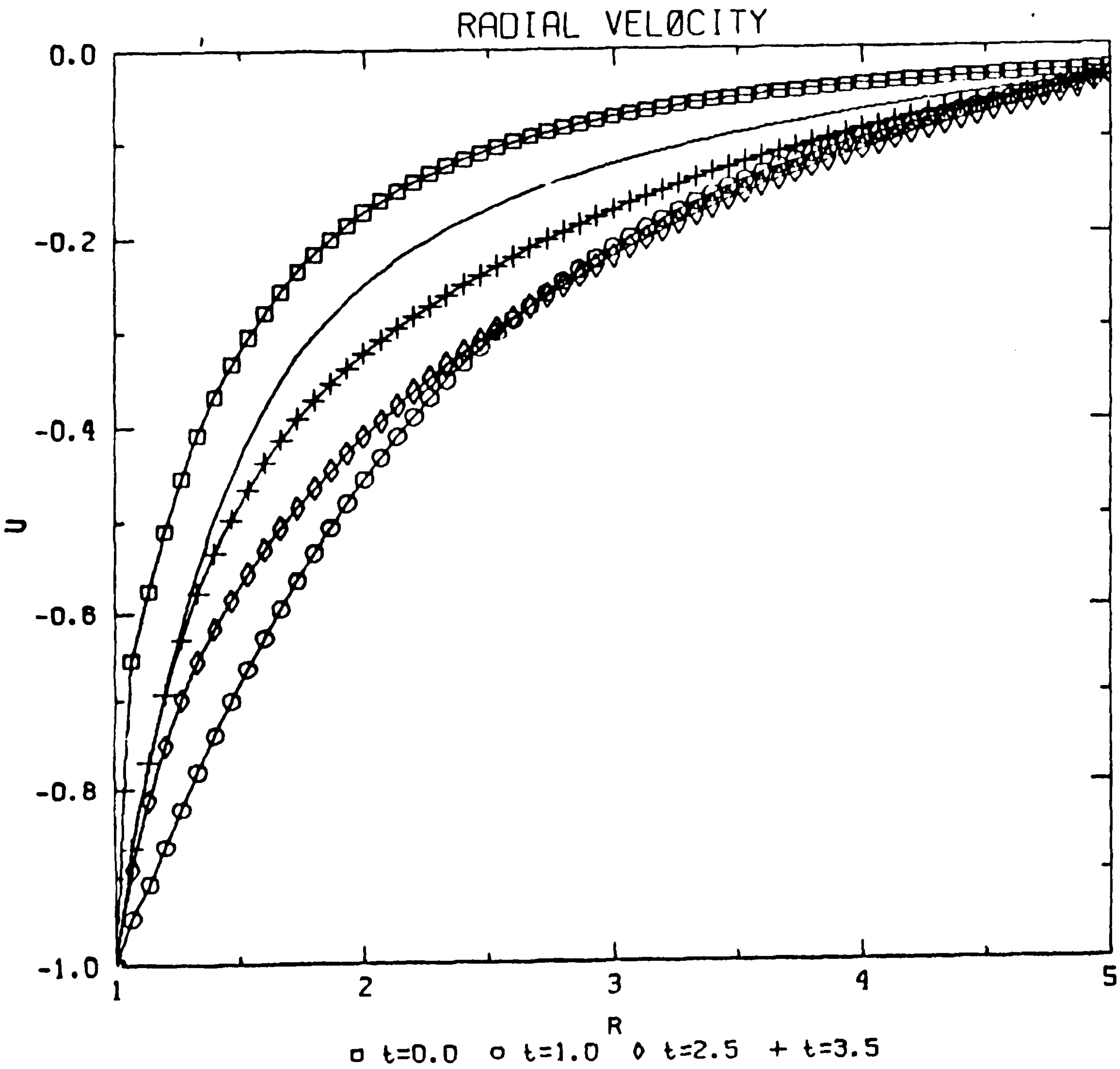

figure 4.1